Turbulence Theory and Modeling

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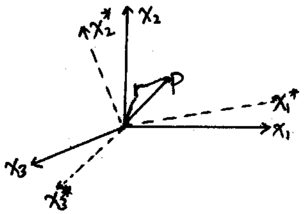
Contents

Definition of Tensor-1

0차 Tensor; A : scalar
1차 Tensor; A_i : vector
2차 Tensor; A_{ij} :
⋮



$$r^2 = x_i \cdot x_i = x_i^* \cdot x_i^* = x_1^2 + x_2^2 + x_3^2 \quad (i=1,\ 2,\ 3) \qquad (1.1)$$

<u>방향여현</u>(Directional cosine):

$$e_{ij} = \cos(x_i,\ x_j^*) \qquad (1.2)$$

$$x_i^* = x_j\, e_{ji} \qquad (1.3)$$

$$x_j = x_i^*\, e_{ij}$$

(e_{ij}는 정규직교좌표계임!!!!!)

$e_{ij}\ e_{kj} = 1$ if $i = k$

$e_{ij}\ e_{kj} = 0$ if $i \neq k$ $\qquad e_{ij}\ e_{kj} = \delta_{ik}$ (1.4)

Proof of Eq.(1.1)

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$= x_j^* x_k^* e_{1j}\ e_{1k} + x_j^* x_k^* e_{2j}\ e_{2k} + x_j^* x_k^* e_{3j}\ e_{3k} = x_j^* x_k^* e_{ij}\ e_{ik} = x_j^* x_j^*$$

Definition of Tensor-2

1차 Tensor A_i 의 정의(Tensor의 변환법칙)

$$A_i^* = A_j\, e_{ji}, \qquad A_j = A_i^*\, e_{ji} \tag{1.5}$$

2차 Tensor는 1차 Tensor의 곱으로

$$C_{ij} = A_i\, B_j \tag{1.6}$$

식(1.5)로부터

$$A_i\, B_j = A_k^*\, e_{ki}\, B_l^*\, e_{lj} = A_k^* B_l^*\, e_{ki}\, e_{lj}$$

$$\therefore\ C_{ij} = C_{kl}^*\, e_{ik}\, e_{jl} \tag{1.7}$$

n차 Tensor는

$$A_{ijk\ldots m}^* = A_{pq\ldots t}\, e_{pi}\, e_{qj} \cdots e_{tm} \tag{1.8}$$

2개의 특수한 Tensor

(i) $$\delta_{ij} = e_{ik}\, e_{kj} = 1 \quad (i=j), \qquad \neq 0 \quad (i \neq j) \tag{1.9}$$

(ii) $$\varepsilon_{ijk} = \begin{cases} 1 & (1,2,3,1,2,3\text{의 순}) \\ -1 & (1,3,2,1,3,2\text{의 순}) \\ 0 & (i,j,k\text{중 두개가 같음}) \end{cases} \tag{1.10}$$

Useful Rules for Tensor

(a) m차, n차 Tensor의 곱은 (m+n)차 Tensor

ex) $A_{ij} B_{lmn} = C_{ijlmn}$ (1.11)

(b) Tensor가 서로 같은 두 개의 첨자를 가진 경우, Tensor의 차수는 2개 감소한다.

ex) $B_{ijjq} = B_{i11q} + B_{i22q} + B_{i33q} = C_{iq}$ (1.12)

(c) Tensor A_{kjrs} 에 δ_{ij} 를 곱하면 A_{kjrs} 의 첨자 j 가 i 로 바뀐다.

ex) $\delta_{ij} A_{kjrs} = e_{im}\, e_{jm}\, e_{kl}\, e_{jn} A^{*}_{lnrs}$

$= e_{im}\, e_{kl} A^{*}_{lmrs} = A_{kirs}$ (1.13)

(d) n차 Tensor의 m차 미계수는 (m+n)차 Tensor이다.

ex) $\frac{\partial}{\partial x_j}(A_i) = \frac{\partial}{\partial x^{*}_{l}} \frac{\partial x^{*}_{l}}{\partial x_j}(A^{*}_{k} e_{ik})$

$= e_{jl} \frac{\partial}{\partial x^{*}_{l}}(A^{*}_{k} e_{ik}) = e_{ik}\, e_{jl}\left(\frac{\partial A^{*}_{k}}{\partial x^{*}_{l}}\right)$ (1.14)

Conservation Laws-1
(Mass Conservation)

기호: $f = \bar{f} + f'$ (2.1)

$\dot{f} = \dfrac{\partial f}{\partial t}$ (2.2)

$f_{,j} = \dfrac{\partial f}{\partial x_j}$ (2.3)

Mass Conservation

$\dot{\rho} + (\rho u_j)_{,j} = 0$ (2.4)

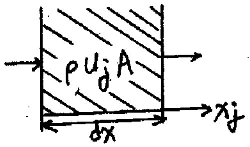
또는 $\dot{\rho} + \rho_{,j} u_j + \rho u_{j,j} = 0$

$\therefore \dfrac{D\rho}{Dt} + \rho u_{j,j} = 0$ (2.4')

$\Leftarrow \dfrac{D\phi}{Dt} = \dot{\phi} + u_j \phi_{,j}$

비압축성 유체 ($\rho = const$)인 경우;

$u_{i,i} = 0 \quad \rightarrow \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ (2.5)

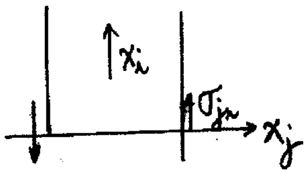


$(\dot{\rho}A) + (\rho u_j A)_{,j} = 0$

$A = \dfrac{\rho}{\rho} = 1$; 연속방정식

Conservation Laws-2
(Momentum Equation)

Momentum Equation



$$A \rightarrow u_i$$

$$(\rho \dot{u}_i) + (\rho u_j u_i)_{,j} = \sigma_{ji,j} + \rho f_i \quad (2.6)$$

표면력 체적력

$$\sigma_{ji} = -p\,\delta_{ij} + \tau_{ij}$$

(stress tensor)

* σ_{ij}는 대칭 Tensor, 즉 $\sigma_{ij} = \sigma_{ji}$
또한 τ_{ij}도 대칭 Tensor, 즉 $\tau_{ij} = \tau_{ji}$

$$p\,\delta_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \qquad \tau_{ij} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Conservation Laws-3
(Energy Equation)

Energy Equation

$$A \rightarrow \quad e_T = e + u_i u_i / 2$$

$$(\dot{\rho e_T}) + (\rho u_j e_T)_{,j} = (\sigma_{ji} u_i)_{,j} - q_{j,j} + \dot{Q} \qquad (2.7)$$

(Work energy) (Heat flux) (발열)

열역학 제 2법칙

$\Sigma = T\dfrac{ds}{dt}$; 단위체적당 Entropy 증가율 (발열 없음)

$$\Sigma = (\dot{\rho s}) + (\rho u_j s)_{,j} + (q_j / T)_{,j} \geq 0 \qquad (2.8)$$

상태방정식

ex) $pv = RT \qquad v = 1/\rho \rightarrow dv = -\dfrac{d\rho}{\rho^2}$

$$\delta Q = Tds = de - pdv = de - (p/\rho^2)d\rho$$

$$T\frac{ds}{dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{De}{Dt} + \frac{p}{\rho} u_{i,i} \qquad (2.9)$$

Conservation Laws-4 (Energy Equation)

운동에너지 방정식 식(2.6)x u_i

$$(\rho u_i u_i/2) + (\rho u_j u_i u_i/2)_{,j} = u_i \sigma_{ji,j} + \rho u_i f_i$$

$$= (u_i \sigma_{ji})_{,j} - u_{i,j} \sigma_{ji} + \rho u_i f_i \qquad (2.10)$$

열에너지 방정식 식(2.7)+식(2.10)

$$(\dot{\rho e}) + (\rho u_j e)_{,j} = u_{i,j} \sigma_{ji} - q_{j,j} + \dot{Q} - \rho u_i f_i \qquad (2.11)$$

(표면력에 의한 일)

Conservation Laws-5
(비가역성)

비가역성 식(2.8)로 부터 식(2.4'), (2.9), (2.11)을 이용해서

$$\Sigma = \dot{\rho} s + s(\rho u_j),_j + \rho \dot{s} + \rho u_j s,_j + (q_j/T),_j$$

$$T\Sigma = \rho T \dot{s} + \rho u_j T s,_j + T(q_j/T),_j$$

$$= \rho\left(\dot{e} - \frac{p}{\rho^2}\dot{\rho}\right) + \rho u_j\left(e,_j - \frac{p}{\rho^2}\rho,_j\right) + T(q_j/T),_j$$

$$= \rho\dot{e} + \rho u_j e,_j - \frac{p}{\rho}\dot{\rho} - u_j\frac{p}{\rho}\rho,_j + T(q_j/T),_j$$

$$= u_{i},_{j}\sigma_{ji} - q_{j},_{j} - \dot{\rho}e - e(\rho u_j),_j$$

$$- \frac{p}{\rho}\dot{\rho} - u_j\frac{p}{\rho}\rho,_j + T(q_j/T),_j$$

$$= u_{i},_{j}\tau_{ij} + \lambda(T,_j)^2/T \geq 0 \qquad (2.12)$$

(점성소산)

(**주의**) 식(2.12)을 유도하는데 있어 발열 및 체적력은 없다고 가정함.
또한, 후술하는 식(2.19) 및 식(2.23)을 적용해야함.

Conservation Laws-6
(Newton 유체의 응력구성방정식)

Newton 유체의 응력구성 방정식

σ_{ij} 는 $u_{i,j}$의 선형적인 함수.

$$u_{i,j}=\frac{1}{2}(u_{i,j}+u_{j,i})+\frac{1}{2}(u_{i,j}-u_{j,i}) \quad (2.14)$$

$$S_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \text{ ; strain tensor}$$

$$\Omega_{ij}=\frac{1}{2}(u_{i,j}-u_{j,i}) \text{ ; retention tensor} \quad (2.15)$$

$\Omega_{ij}=-\Omega_{ji}$ 이고, $\sigma_{ij}=\sigma_{ji}$ 이므로

$$\therefore \sigma_{ij}=func(S_{ij}) \quad (2.16)$$

S_{ij} 에 대한 가장 일반적인 선형함수는

$$\sigma_{ij}=A\delta_{ij}+BS_{ij} \quad (2.17)$$

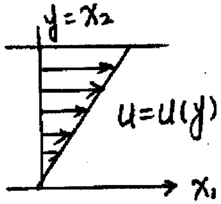
$\frac{1}{3}\sigma_{ii}=average\ \ normal\ \ stresses=-p$ 이므로

$$-p=\frac{1}{3}A\delta_{ii}+\frac{1}{3}BS_{ii}$$

$$\rightarrow A=-p-\frac{B}{3}S_{kk}$$

Conservation Laws-7
(Newton 유체의 응력구성방정식)

$$\therefore \ \sigma_{ij} = \left(-p - \frac{B}{3} S_{kk} \right) \delta_{ij} + B S_{ij} \tag{2.18}$$



$$\frac{du}{dy} = \frac{du}{dx_2}$$

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} 0 & \frac{du}{dy} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\frac{du}{dy} & 0 \\ \frac{1}{2}\frac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{21} = \mu \frac{du}{dy} = \left(-p - \frac{B}{3} \times 0 \right) \cdot 0 + B \cdot \frac{1}{2} \frac{du}{dy}$$

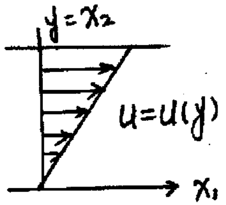
$$\therefore B = 2\mu$$

$$\therefore \ \sigma_{ij} = \left(-p - \frac{2\mu}{3} S_{kk} \right) \delta_{ij} + 2\mu S_{ij} \tag{2.19}$$

Conservation Laws-7
(Newton 유체의 응력구성방정식)

$$\rightarrow A = -p - \frac{B}{3} S_{kk}$$

$$\therefore \sigma_{ij} = \left(-p - \frac{B}{3} S_{kk}\right)\delta_{ij} + B S_{ij} \qquad (2.18)$$



$$\frac{du}{dy} = \frac{du}{dx_2}$$

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} 0 & \frac{du}{dy} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\frac{du}{dy} & 0 \\ \frac{1}{2}\frac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{21} = \mu\frac{du}{dy} = \left(-p - \frac{B}{3}\times 0\right)\cdot 0 + B\cdot\frac{1}{2}\frac{du}{dy}$$

$$\therefore B = 2\mu$$

$$\therefore \sigma_{ij} = \left(-p - \frac{2\mu}{3} S_{kk}\right)\delta_{ij} + 2\mu S_{ij} = -p\,\delta_{ij} + \tau_{ij} \qquad (2.19)$$

Conservation Laws-8
(Newton 유체의 응력구성방정식)

* $\mu = const.$ 인 경우

$$\sigma_{ji,j} = \left[\left(-p - \frac{2\mu}{3} S_{kk}\right)\delta_{ij}\right]_{,j} + 2\mu S_{ij,j}$$

$$= -p_{,i} + \mu u_{i,jj} + \frac{1}{3}\mu u_{j,ji} \qquad (2.20)$$

또한, $\rho = const.$ 일 때, ($u_{j,j} = 0$)

$$\sigma_{ji,j} = -p_{,i} + \mu u_{i,jj} \qquad (2.21)$$

식(2.6)의 운동량 방정식은

$$(\rho \dot{u}_i) + (\rho u_j u_i)_{,j} = -p_{,i} + \mu u_{i,jj} + \rho f_i \qquad (2.22)$$

Conservation Laws-9
(열에너지방정식)

열에너지 방정식

Fourier 법칙 (열전도): $q_i = -\lambda T_{,i}$ (2.23)

표면력에 의한 일

$$u_{i,j}\sigma_{ji} = u_{i,j}\left[\left(-p-\frac{2\mu}{3}S_{kk}\right)\delta_{ij}+2\mu S_{ij}\right]$$

$$= -p\,u_{j,j} - \frac{2}{3}\mu\,u_{i,i}\,u_{j,j} + \mu\,u_{i,j}(u_{i,j}+u_{j,i}) \qquad (2.24)$$

점성 소산율 Φ;

$$\Phi = -\frac{2}{3}u_{i,i}\,u_{j,j} + u_{i,j}(u_{i,j}+u_{j,i}) \qquad (2.25)$$

식(2.11)로부터

$$\dot{(\rho e)} + (\rho u_j e)_{,j} = -p\,u_{j,j} + (\lambda T_{,j})_{,j} + \mu\Phi + \dot{Q} - \rho u_i f_i \qquad (2.26)$$

엔탈피: $h = e + p/\rho \rightarrow e = h - p/\rho$

$$\dot{(\rho h - p)} + (\rho h u_j - p u_j)_{,j} = \dot{(\rho h)} - \dot{p} + (\rho h u_j)_{,j} - (p u_j)_{,j}$$

$$\therefore \dot{(\rho h)} + (\rho h u_j)_{,j} = (\lambda T_{,j})_{,j} + \mu\Phi + \underbrace{\dot{p} + u_j p_{,j}}_{\frac{Dp}{Dt}} + \dot{Q} - \rho u_j f_j \qquad (2.27)$$

Conservation Laws-10
(열에너지방정식)

또한,

$$dh = C_p dT + \frac{1}{\rho}(1-\beta T)dp$$

$$\left(\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \text{ 을 이용해서} \right)$$

<u>에너지방정식</u>(유체의 온도를 기술함)

$$\rho C_p \dot{T} + \rho C_p u_j T_{,j} = (\lambda T_{,j})_{,j} + \mu\Phi + \beta T(\dot{p} + u_j p_{,j}) + \dot{Q} - \rho u_j f_j \qquad (2.30)$$

$$\because dh = Tds + dp/\rho, \quad ds = \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp = \left(\frac{\partial s}{\partial T}\right)_p dT - (\beta/\rho)dp$$

* 이상기체에 대해서는 ($\beta = 1/T$, $\lambda = const.$로 가정)

$$\rho C_p \dot{T} + \rho C_p u_j T_{,j} = (\lambda T_{,j})_{,j} + \mu\Phi + (\dot{p} + u_j p_{,j}) + \dot{Q} - \rho u_j f_j \qquad (2.31)$$

(소산) (압력일)

* 비압축성 유체 ($\beta = 0$)

$$\dot{T} + u_j T_{,j} = \kappa T_{,jj} + \frac{\nu}{C_p}\Phi + \frac{\dot{Q}}{\rho C_p} - \frac{1}{C_p} u_j f_j \qquad (2.32)$$

$Ma \le 0.3$이고, 체적력이 없는 경우,

$$\dot{T} + (u_j T)_{,j} = \frac{DT}{Dt} = \kappa T_{,jj} \qquad (2.33)$$

➡ T 에 대해서 선형임!!!!!!