Hydrostatics

HYDROSTATICS

No motion ⤳ Zero deformation ⤳

Hence zero shear stress ⤳

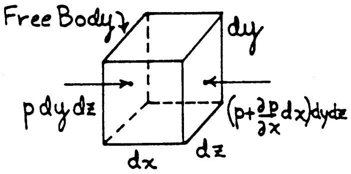
Hence hydro static state of stress

$p = p(x, y, z)$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

NET FORCE DUE TO p-FIELD

Free Body



$$(\Sigma F_x)_{p\text{-Field}} = \left(-\frac{\partial p}{\partial x}dx\right)dydz$$

$$(\Sigma \mathbb{F})_{p\text{-field}} = (-\nabla p)\,dxdydz$$

$$\nabla p \equiv i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y} + k\frac{\partial p}{\partial z}$$

N.B. ∇p vs. p

BODY FORCE:

$\mathbb{G}$ per unit Mass

$$(\Sigma \mathbb{F})_{body} = \mathbb{G}\,(\rho\, dxdydz)$$

STATIC EQUIL.

$$-\nabla p + \rho\mathbb{G} = 0; \quad \boxed{\nabla p = \rho\mathbb{G}}$$

TRANSP. Isobars ⊥ $\mathbb{G}$-field

HYDROSTATICS IN A LOCAL GRAVITY FIELD

z ↑ ↓ g   EARTH

$$\mathbb{G} = i(0) + j(0) + k(-g)$$

$$\nabla p = i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y} + k\frac{\partial p}{\partial z}$$

$$\therefore \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0; \quad \frac{\partial p}{\partial z} = \boxed{\frac{dp}{dz} = -\rho g}$$

"INCOMPRESSIBLE" FLUIDS

$dp = -\rho g\, dz$; ρ = const.

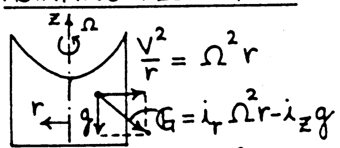
$p + \rho g z$ = CONST; $\Delta p = -\rho g \Delta z$

TRANSP. Manometers

Buoyancy

Forces on surfaces

ROTATING FLUID MASS



$$\frac{V^2}{r} = \Omega^2 r$$

$$\mathbb{G} = i_r\Omega^2 r - i_z g$$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r; \quad \frac{\partial p}{\partial z} = -\rho g$$

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$$

$$= \rho\Omega^2 r\, dr - \rho g\, dz$$

$$p - p_0 = \frac{\rho\Omega^2}{2}(r^2 - r_0^2) - \rho g(z - z_0)$$

Surf. of const p = parab. of rev.

HYDROSTATICS OF COMPRESSIBLE FLUID

(a) Earth's atmosphere

$$dp/dz = -\rho(p,T)g$$

$$\int_{p_0}^{p}\frac{dp}{\rho(p,T)} = -\int_0^z g\, dz = -gz$$

$$\rho(p,T) = ?$$

If isothermal P.G.: $\rho = p/RT$

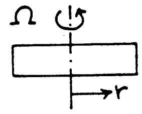
$$\int_{p_0}^{p}\frac{dp}{\rho} = RT\int_{p_0}^{p}\frac{dp}{p} = RT\ln\frac{p}{p_0}$$

$$\frac{p}{p_0} = e^{-gz/RT}$$

If $T \cong 278°K$ (~40°F)

$p/p_0 = 1/e$ at $z \cong 27{,}000$ ft.

(b) Gas Centrifuge



Neglect gravity

$$\frac{\partial p}{\partial r} = \frac{dp}{dr} = \rho\Omega^2 r$$

Assume: P.G.; isothermal: $\rho = \frac{p}{RT}$

$$\int_{p_0}^{p}\frac{dp}{p} = \frac{\Omega^2}{RT}\int_0^r r\,dr$$

$$p/p_0 = e^{\Omega^2 r^2/2RT}$$

N.B. $c^2 = kRT$

EQUILIBRIUM AGAIN, FOR MOMENTS

$$\nabla p = \rho\mathbb{G} \qquad (\Sigma\mathbb{F} = 0)$$

Perform $\nabla\times\nabla p = \nabla\times(\rho\mathbb{G})$

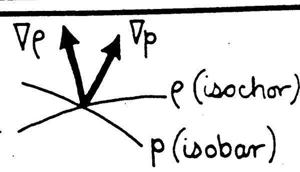
$$\nabla\times\nabla p = \rho(\nabla\times\mathbb{G}) + \nabla\rho\times\mathbb{G}$$

But $\nabla\times\nabla p = 0$

$$\therefore \nabla\rho\times\mathbb{G} = -\rho(\nabla\times\mathbb{G})$$

Hence $\nabla\rho\times\nabla p = \nabla\rho\times\rho\mathbb{G}$

$$= -\rho^2(\nabla\times\mathbb{G})$$



TRANSP. Types of Forces

FREE SURFACE IF $\nabla\times\mathbb{G} = 0$

Free Surf ⤳ const. ρ

∴ const. p

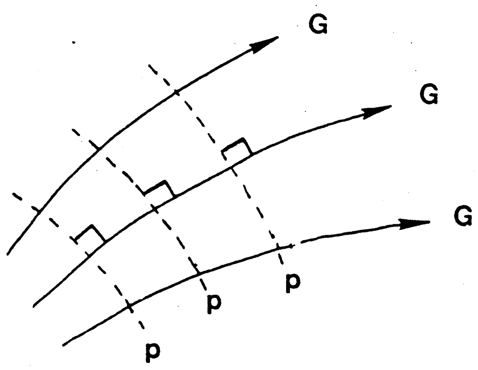
But p-surf. ⊥ $\mathbb{G}$

Hence ρ-surface ⊥ $\mathbb{G}$

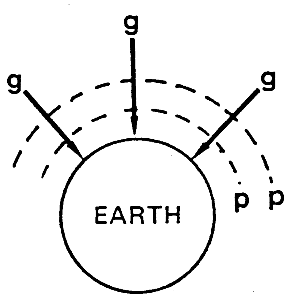
TRANSP. Examples

(a) Water level; (b) Rotating Mass

ISOBARS ARE NORMAL TO G-FIELD



ISOBARS ARE NORMAL TO G-FIELD



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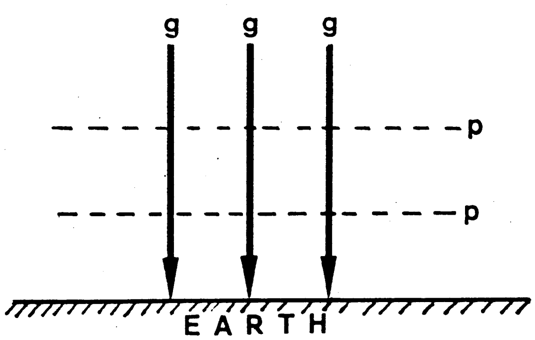

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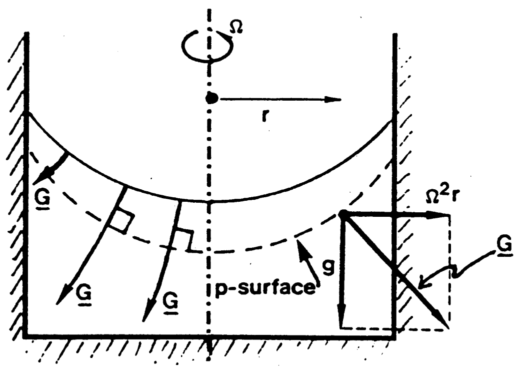
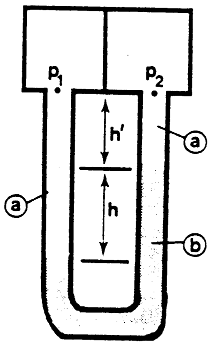




MANOMETERS

$$p_1 + \rho_a g(h + h') = p_2 + \rho_a g h' + \rho_b g h$$

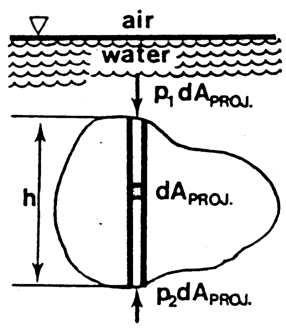
$$p_1 - p_2 = (\rho_b - \rho_a) g h$$

$$= (1 - \rho_a/\rho_b)\rho_b g h$$

SPECIAL FORMS

- Barometer
- Two-Fluid Manometer
- Inclined Manometer

BUOYANCY

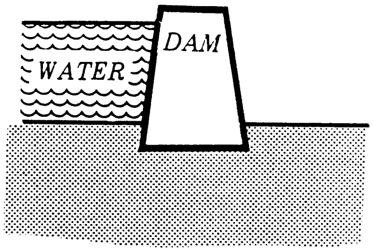


NET UPWARD FORCE

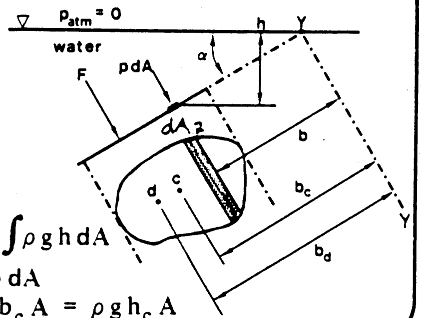
$$= \int (p_2 - p_1)\, dA_{PROJ.}$$

$$= \int \rho g h\, dA_{PROJ.}$$

$$= \rho g \forall \quad \text{(Archimedes)}$$



FORCES AND MOMENTS ON PLANE SURFACES



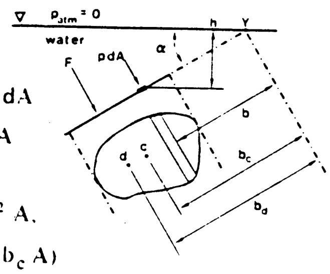
$$F = \int p\,dA = \int \rho g h\,dA$$
$$= \rho g \sin\alpha \int b\,dA$$
$$= (\rho g \sin\alpha)\, b_c A = \rho g h_c A$$

FORCES AND MOMENTS ON PLANE SURFACES

$$b_d F = \int b\,p\,dA$$
$$= \int b\,(\rho g b \sin\alpha)\,dA$$
$$= \rho g \sin\alpha \int b^2\,dA$$
$$= \rho g I_{YY} \sin\alpha$$

Since $I_{YY} = I_c + b_c^2 A$,

$$b_d - b_c = I_c/(b_c A)$$



CONSERVATIVE BODY FORCES	NON-CONSERVATIVE BODY FORCES
$\nabla \times \underline{G} = 0$	$\nabla \times \underline{G} \neq 0$
Isobars and isochors do coincide.	Isobars and isochors do not coincide.
EXAMPLES:	EXAMPLES:
gravity electrostatic magnetostatic centrifugal	Lorentz ($\underline{J} \times \underline{B}$) coriolis